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The Orthomodular Poset of Projections of A Symmetric Lattices

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In a Hilbert space, there exists a natural correspondence between continuous projections and particular pairs of closed subspaces. In this paper, we generalize this situation and associate to a symmetric lattice L a subset $P(L)$ of $L \times L$, called its projection poset. If L is the lattice of closed subspaces of a topological vector space then elements of $P(L)$ correspond to continuous projections and we prove that automorphisms of $P(L)$ are determined by automorphisms of the lattice L when this lattice satisfies some basic properties of lattices of closed subspaces.

KEY WORDS: orthomodular lattices; symmetric lattices; lattices of subspaces.
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1. INTRODUCTION

If H is a Hilbert space then there exists a natural correspondence between orthogonal projections defined on H and closed subspaces of H: to every projection p is associated its image Im $p = p(H)$. Now if p is a not necessarily orthogonal projection (called a projection or a linear projection in the sequel) then p is determined by the pair (Im p, Ker p) of closed subspaces; and so there exists a bijection between the set of all projections and a subset $P(\mathcal{C}(H))$ of $\mathcal{C}(H) \times \mathcal{C}(H)$, where $\mathcal{C}(H)$ denotes the orthomodular lattice (abbreviated OML) of closed subspaces of H.

A purpose of this paper is to generalize the above situation and associate to certain lattices L a subset $P(L)$ of $L \times L$ which is called its projection poset. When L is the lattice of all subspaces of a vector space E then $P(L)$ is isomorphic to the poset of all linear projections defined on E and we obtain a continuous version of this result which can be, roughly speaking, stated as follows: if L is the lattice of closed subspaces of a topological vector space E then $P(L)$ is isomorphic to the poset of continuous linear projections defined on E.

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An origin of this work is a paper by Ovchinnikov (1993) in which the author determines the automorphism group of the poset of projections defined on a Hilbert space by means of automorphisms and antiautomorphisms of the lattice of all closed subspaces of H. We prove, in a second part of the paper, that this result can be extended to an abstract projection poset $P(L)$ if L is a lattice satisfying some basic properties of lattices of closed subspaces.

Information about the lattice concepts used in this paper may be found in K othe (1969), Maeda and Maeda (1970) is a good reference for topological vector spaces.

2. THE ORTHOMODULAR POSET OF PROJECTIONS OF A LATTICE

2.1. Definition and Structure

Let a and b be two elements of a lattice L. We say that (a, b) is a modular pair, and write $(a, b)M$, when

$$(x \vee a) \wedge b = x \vee (a \wedge b) \quad \text{for every} \quad x \leq b.$$

If $(a, b)M$ holds in the dual lattice L^* of L, we say that (a, b) is a dual-modular pair and write $(a, b)M^*$.

A lattice L is said to be M-symmetric if $(a, b)M$ implies $(b, a)M$ and M^*-symmetric if L^* is M-symmetric. A symmetric lattice is lattice which is a M-symmetric and M^*-symmetric lattice.

Let L be a symmetric lattice with 0 and 1. The direct product of L and L^* is also a lattice and the set of all elements (a, b) of $L \times L^*$ such that in the lattice L,

$$a \vee b = 1, \quad a \wedge b = 0, \quad (a, b)M, \quad (a, b)M^*$$

is called the projection poset of L. This poset is denoted by $P(L)$ and any element of $P(L)$ is called a projection of L.

If (a, b) is a projection of L then, as L is a symmetric lattice, (b, a) is also a projection and we write $(b, a) = (a, b)^\perp$. The first claim of the following proposition is proved in Mushtari (1998). It is also a consequence of more general results of Harding (1996).

Proposition 1. *If L is a symmetric lattice with 0 and 1 then* $(P(L), \leq, \perp)$ *is an orthomodular poset (abbreviated OMP). If L possesses a structure of OMP then this OMP is naturally isomorphic to a suborthomodular poset of* $P(L)$.

Proof: It is clear that $(0, 1)$ is the least element of $P(L)$ and $(1, 0)$ the greatest element. It is also evident that $(a, b)^{\perp\perp} = (a, b)$ and that $(a, b) \leq (c, d)$ implies $(c, d)^\perp \leq (a, b)^\perp$.

For two projections (a, b) and (c, d), we write $(a, b) \perp (c, d)$ if $(a, b) \leq (c, d)^\perp$ and we must prove that $(a, b) \perp (c, d)$ implies that $(a, b) \vee (c, d)$ exists.

As in $L \times L^*$, $(a, b) \vee (c, d) = (a \vee c, b \wedge d)$, it suffices to prove that $(a \vee c, b \wedge d) \in P(L)$. This result will be an easy consequence of the following technical lemma. □

Lemma 1. *If $a \neq 0,\ b,\ c,\ d \neq 1$ are different elements of a bounded lattice L such that:*

(a) $a \wedge b = c \wedge d = 0, a \vee b = c \vee d = 1, a \leq d, b \geq c$;
(b) $(a, b)M, (b, a)M^*, (c, d)M, (d, c)M^*$

then

1. $\mathcal{B} = \{0, 1, a, b, c, d, b \wedge d, a \vee c\}$ *is a distributive sublattice of L which is a* 2^3*-element Boolean lattice;*
2. $(a, c)M,\ (c, a)M,\ (a, b \wedge d)M,\ (b \wedge d, a)M,\ (c, b \wedge d)M,\ b \wedge d, c)M,\ (a \vee c, d)M, (a \vee c, b)M, (d, b)M, (c, d)M, (a \vee c, b \wedge d)M;$
3. $(a, c)M^*,\ (c, a)M^*,\ (b \wedge d, a)M^*,\ (b \wedge d, c)M^*,\ (a \vee c, b)M^*,\ (b, a \vee c)M^*,\ (a \vee c, d)M^*,\ (d, a \vee c)M^*,\ (d, b)M^*,\ (b, d)M^*,\ (b \wedge d, a \vee c)M^*.$

If moreover, L is a symmetric lattice then $(x, y)M$ and $(x, y)M^*$ hold in L for every pair (x, y) of elements of $\mathcal{B}$.

proof:

1. By $(d, c)M^*$ and $c \leq b$ we have

$$(a \vee c) \vee (b \wedge d) = a \vee [c \vee (b \wedge d)]$$
$$= a \vee [b \wedge (c \vee d)] = a \vee (b \wedge 1) = a \vee b = 1.$$

Similarly, $(a, b)M$ and $c \leq b$ yield $(a \vee c) \wedge (b \wedge d) = 0$. By using $(c, d)M$ and $a \leq d$, we have:

$$(a \vee c) \wedge d = a \vee (c \wedge d) = a$$

and the proofs of $b = (b \wedge d) \vee c$, $d = (b \wedge d) \vee a$, and $c = (a \vee c) \wedge b$ are similar.

Now, it is easy to check that $\mathcal{B} = \{0, 1, a, b, c, d, b \wedge c, a \vee d\}$ is a 2^3 element sublattice of L. In this lattice, $\{a, c, b \wedge d\}$ is the set of all atoms, $\{d, b, a \vee c\}$ the set of all coatoms and $\mathcal{B}$ can be equipped of a structure of Boolean lattice.

2. Many proofs of (2) are similar. We will give only an example.

Let $x \leq b \wedge d$. By using $(c, d)M$, $x \vee a \leq d$ and $(a, b)M$, $x \leq b$, we can write:

$$(x \vee a \vee c) \wedge (b \wedge d) = [(x \vee a \vee c) \wedge d] \wedge b = [(x \vee a) \vee (c \wedge d)] \wedge b$$

$$= (x \vee a) \wedge b = x \vee (a \wedge b) = x$$
$$= x \vee [(a \vee c) \wedge (b \wedge d)]$$

and thus $(a \vee c, b \wedge d)M$ holds.

3. It suffices to apply the results of (2) in L^*.

Now we return to the proof of the proposition and prove that $P(L)$ satisfies the orthomodular law. Assume that, for two projections (a, b) and (c, d) we have $(a, b) \leq (c, d)$, that is $a \leq c$ and $b \geq d$. Using Lemma 1, we have

$$(a, b) \vee [(a, b)^{\perp} \wedge (c, d)] = (a, b) \vee [(b, a) \wedge (c, d)] = (a, b) \vee (b \wedge c, a \vee d)$$
$$= (a \vee (b \wedge c), b \wedge (a \vee d)) = (c, d)$$

and the orthomodular law is satisfied.

If L is an OMP for an orthocomplementation $x \mapsto x^{\vdash}$, consider the mapping $\varphi : L \mapsto L \times L^*$ defined by $\varphi(x) = (x, x^{\vdash})$. Since L is an OMP, $(a, a^{\vdash})M$ and $(a, a^{\vdash})M^*$ hold for every $a \in L$ and thus $\varphi(L) \subset P(L)$. We have $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$ and $\varphi(a^{\vdash}) = (a^{\vdash}, a) = (a, a^{\vdash})^{\perp} = \varphi(a)^{\perp}$. Therefore, the OMP L is isomorphic to $\varphi(L)$.

The proof is complete.

2.2. Examples

1. Let E be a vector space. The lattice L of all subspaces of E is a complete modular lattice and $P(L) = \{(F, G) \in L^2 \mid F \oplus G = E\}$. An element (F, G) of $P(L)$ can be identified with the linear projection $p : E \mapsto E$ such that $\operatorname{Im} p = F$ and $\operatorname{Ker} p = G$ and the set of all linear projections of the vector space E is an OMP naturally isomorphic isomorphic to $P(L)$ by the correspondence $p \mapsto (\operatorname{Im} p, \operatorname{Ker} p)$.
2. For $n \in \mathbb{N}$, $n \geq 3$, let L_n be the lattice of all subspaces of the vector space $\mathbb{C}^n$. It is proved in Chevalier (2000) that there exists $2^{2^{\aleph_0}}$ nonisomorphic structures of orthomodular lattices on L_n and therefore there exists $2^{2^{\aleph_0}}$ nonisomorphic suborthomodular posets in the projection lattice $P(L_n)$. Each of them has an underlying lattice isomorphic to L_n.
3. Let L be a complete modular complemented lattice which is upper-continuous (For example, L is a continuous geometry or the lattice of Example 1). It is proved in Chevalier (1999) that $P(L)$ is an OMP satisfying the maximality principle. This means that every generalized Sasaki projection is non empty. In such OMPs, many

properties of ideals generalize properties of ideals in OMLs (Chevalier, 1999).

Remark. Our purpose, in the definition of the projection poset $P(L)$ of a lattice L, is to determine when L is the lattice of all closed subspaces of a topological vector space E, a poset isomorphic to the poset of all continuous linear projections defined on E. As a continuous linear projection is determined by an ordered decomposition of E in a topological direct sum of two closed subspaces, the conditions imposed in the definition of $P(L)$ can be justified as follows:

1. On the set of all projections of a vector space, and more generally on the set of all idempotents of a unitary ring, there exist an usual order relation and an usual orthocomplementation which are defined as follows. If p and q are two linear projections then $p \leq q$ means $p = qp = pq$ and is equivalent to Im $p \subset$ Im q, Ker $p \supset$ Ker q. On the other hand, $p^{\perp} = 1 - p$ and Im $(1 - p) =$ Ker p, Ker$(1 - p) =$ Im p. These properties justify that $P(L)$ is $P(L)$ is considered as a subset of the ordered set $L \times L^*$ and the definition $(a, b)^{\perp} = (b, a)$.
2. Condition $a \wedge b = 0$ is clear since, in lattices of closed subspaces, $a \wedge b = a \cap b$.
3. $a \vee b = 1$ and $(a, b)M^*$ is, for subspaces, generally equivalent to $a + b = 1$ (Mackey, 1945, Theorem III-6, Maeda and Maeda, 1970, Lemma 31.1, Holland, 1964, Theorem 2).

   Therefore, the previous two conditions force $a \oplus b = 1$ if a and b are subspaces.
4. Many lattices of closed subspaces are symmetric (Mackey, 1945, Corollary 4 of Theorem III-6 and Corollary 2 of Theorem III-7). Another motivation is the fact that if $(a, b) =$ (Ker p, Im p) then $(b, a) =$ (Im $(1 - p)$, Ker $(1 - p)$) and so the two pairs (a, b) (b, a) fulfill the same properties related to modularity.
5. Condition $(a, b)M$ is the least obvious of all. We will see that this condition forces the direct sum $a \oplus b$ to be a topological sum or, equivalently, the continuity of the linear projection having a as range and b as kernel.

An algebraic setting for the lattice properties of the set of all closed subspaces of a topological vector space is the concept of DAC-lattice and DAC-lattice are closely related to lattices of closed subspaces of pairs of dual spaces. We will begin the next number by introducing these two concepts.

3. DUAL SPACES AND DAC-LATTICES

An AC-lattice is an atomistic lattice with the covering property: if p is an atom and $a \wedge p = 0$ then $a \lessdot a \vee p$ that is $a \leq x \leq a \vee p$ implies $a = x$ or $a \vee p = x$. If L and L^* are AC-lattices, L is called a DAC-lattice. Any DAC-lattice is symmetric (Maeda and Maeda, 1970, Theorem 27-6)

Irreducible complete DAC-lattices of length ≥ 4 are representable by lattices of closed subspaces and many lattices of subspaces are DAC-lattices (Maeda and Maeda, 1970, Lemma (31.1)). We recall some definitions.

Let E and F be vector spaces over a field K. If there exists a nondegenerate bilinear form $\mathcal{B}$ on $E \times F$, we say that (E, F) is a pair of dual spaces. For a subspace A of E we put

$$A^{\perp} = \{y \in F \mid \mathcal{B}(x, y) = 0 \text{ for every } x \in A\}.$$

Similarly, let

$$B^{\perp} = \{x \in E \mid \mathcal{B}(x, y) = 0 \text{ for every } y \in B\}$$

for every subspace B of F. A subspace A of E is called F-closed if $A = A^{\perp\perp}$ and the set of all F-closed subspaces, denoted by $L_F(E)$ and ordered by set-inclusion, is a complete irreducible DAC-lattice (Maeda and Maeda, 1970, Theorem (33.4)). Conversely, for any irreducible complete DAC-lattice L of length ≥ 4, there exists a pair (E, F) of dual spaces such that L is isomorphic to the lattice of all F-closed subspaces of E (Maeda and Maeda, 1970, Theorem 33.7).

The set of all E-closed subspaces of F is similarly defined and is also a DAC-lattice. The correspondence $A \mapsto A^{\perp}$ is an antiisomorphism between the lattice of F-closed subspaces of E and the lattice of E-closed subspaces of F.

The notion of pair of dual spaces is closely related to the concept of linear system introduced by Mackey (1945). Let E be a real vector space and let X be a subspace of the algebraic dual E^* of E. The pair (E, X) is called a linear system; and, if $l(x) = 0$ for all $l \in X$ implies $x = 0$ then X is said to be total; and (E, X), equipped with the bilinear form $\langle x, l \rangle = l(x)$, $x \in E$, $l \in X$, is naturally a pair of dual spaces. The definition of a linear system extends to vector spaces over arbitrary fields and if (E, F) is a pair of dual spaces for a bilinear form $\langle\ ,\ \rangle$ then, since the form is nondegenerate, F is isomorphic to a subspace E^*_F of the algebraic dual of E and the pair of dual spaces (E, F) may be identified with the linear system (E, E^*_F); and after this identification the bilinear form is the natural one, $(x, l) \in E \times E^*_F \mapsto l(x)$.

In particular, for every vector space E, (E, E^*) is naturally a pair of dual spaces and $L_{E^*}(E)$ coincide with the set of all subspaces of E.

Remarks and examples. (1) If E is a locally convex space then the lattice of all closed subspaces of E is a DAC-lattice (Maeda and Maeda, 1970, Theorem 31.10); and if the lattice of all closed subspaces of a Hausdorff topological vector space

F over $\mathbb{R}$ or $\mathbb{C}$ is a DAC-lattice then there exists a locally convex topology on F leading to the same lattice of closed subspaces (Maeda and Maeda, 1970, Theorem 31.12).

For real or complex vector spaces, the example of the lattice of all subspaces is not different from the example of the lattice of all closed subspaces of a locally convex space since on every real or complex vector spaces there exists a locally convex topology for which every subspace is closed and every linear projection is continuous (Schaefer, 1966, Example page 36 and Exercise 7 page 69).

(2) Let H be an inner product space for an inner product $\langle ., . \rangle$. The pair (H, H) is a pair of dual spaces but there exists another interesting pair of dual spaces naturally associated to H, namely the linear system (H, H'), where H' is the topological dual of H. If H is not complete then H may be identified with a proper subspace of H'. If H is a Hilbert space then these two pairs of dual spaces coincide.

Now we can consider the following problem: if L is the DAC-lattice of F-closed subspaces of a pair of dual spaces (E, F) or the DAC-lattice of closed subspaces of a locally convex vector space, characterize elements of $P(L)$ by means of linear projections of the vector space E. Roughly speaking, the answer is lattice projections correspond to linear continuous projections. Note the difference between the two cases: in one case a topology is given and in the other case we must define a topology.

3.1. The case of a pair of dual spaces

Let E be a vector space over a field K equipped with the discrete topology. A vector space topology on E is said to be a linear space topology (in the sense of Lefschetz) if 0 has a basis of neighborhoods consisting of subspaces.

Now let (E, F) be a pair of dual spaces. The weak linear topology on E, noted $\sigma(E, F)$, is the linear topology defined by taking $\{G^{\perp} \mid G \subset F, \ \dim G < \infty\}$ as a basis of neighborhoods of 0. If F is interpreted as a subspace of the algebraic dual of E then a subbasis of neighborhoods of 0 is formed by kernels of elements of F.

The weak linear topology on F, noted $\sigma(F, E)$, is defined in the same way. These topologies are Hausdorff and are the coarsest topologies that render the map $(x, y) \in E \times F \mapsto \langle x, y \rangle \in K$ separately continuous. Conversely, for any continuous linear form f on E, there exists $y \in F$ such that $f(x) = \langle x, y \rangle$ and for any continuous form g on F, there exists $x \in E$ satisfying $g(y) = \langle x, y \rangle$. Thus F can be interpreted as the topological dual of E for the $\sigma(E, F)$ topology and E as the topological dual of F for the $\sigma(F, E)$ topology.

Moreover, for a subspace $G \subset E$, we have $\overline{G} = G^{\perp\perp}$ and thus to be a closed subspace is unambiguous (see Kothe (1969), Section 10, for information about the weak linear topology).

In the particular case $F = E^*$, every subspace of E is $\sigma(E, E^*)$-closed and every linear mapping $f : E \mapsto E$ is $\sigma(E, E^*)$-continuous.

The following lemma is a step in the characterization of modular pairs in the lattice of closed subspaces of a linear system by Mackey (Mackey, 1945, p. 167)

Lemma 2. *Let (E, F) be a pair of dual spaces. For two closed subspaces X and Y, such that $X \cap Y = \{0\}$ and $X + Y = E$, the following statements are equivalent.*

1. *(X, Y) is a modular pair in the lattices of closed subspaces of E;*
2. $X^{\perp} + Y^{\perp} = F$;
3. *For every $(f, g) \in F^2$ there exists $h \in F$ such that $f(x) = h(x)$ for all $x \in X$ and $g(y) = h(y)$ for all $y \in Y$.*

Proof: (1) ⇔ (2). Condition (1) is equivalent to $(X^{\perp}, Y^{\perp})M^*$ in the lattice of closed subspaces of F. By using (Maeda and Maeda, 1970), Theorem (33.4), we have $(X^{\perp}, Y^{\perp})M^*$ if and only if $X^{\perp} + Y^{\perp} = X^{\perp} \vee Y^{\perp}$ and thus:

$$X^{\perp} + Y^{\perp} = X^{\perp} \vee Y^{\perp} = (X \wedge Y)^{\perp} = (X \cap Y)^{\perp} = \{0\}^{\perp} = F$$

Conversely, $X^{\perp} + Y^{\perp} = F$ implies $X^{\perp} + Y^{\perp} = X^{\perp} \vee Y^{\perp}$.

(2) ⇒ (3). By $X^{\perp} + Y^{\perp} = F$, we have $f - g = h_1 + h_2, h_1 \in X^{\perp}$ and $h_2 \in Y^{\perp}$. If $h = f - h_1 = g + h_2$ then, as $h_1(x) = 0$ for $x \in X$, $f(x) = h(x)$ for all $x \in X$ and, similarly, $g(y) = h(y)$ for all $y \in Y$.

(3) ⇒ (2). Let $f \in F$. By applying 3) to the pairs $(0, f)$ and $(f, 0)$, there exist h_1, $h_2 \in F$ such that $f(x) = h_1(x)$, $0 = h_2(x)$ for all $x \in X$ and $0 = h_1(y)$, $f(y) = h_2(y)$ for all $y \in Y$. We have $f = h_1 + h_2$ with $h_1 \in X^{\perp}$ and $h_2 \in Y^{\perp}$. □

If (E, F) is a pair of dual spaces and X, Y are two closed subspaces of E, we can consider on $X \times Y$ the product topology of the $\sigma(E, F)$ topology. If $X \cap Y = \{0\}$ and $X + Y = E$ then the mapping $T : X \times Y \mapsto E$ defined by $T(x, y) = x + y$ is a continuous bijection and is open if and only if the linear projection p such that $\operatorname{Im} p = X$ and $\operatorname{Ker} p = Y$ is continuous (note that $T^{-1} = (p, 1 - p)$).

The next lemma characterizes modular pairs by means of T. See Maeda and Maeda (1970), Lemma (32.9), for a similar result in locally convex vector spaces.

Lemma 3. *Let (E, F) be a pair of dual spaces and X, Y two closed subspaces of E such that $X \cap Y = \{0\}$ and $X + Y = E$. The pair (X, Y) is modular in the lattice of closed subspaces of E if and only if for every continuous form f defined on $X \times Y$ there exists $g \in F$ satisfying $f = g \circ T$.*

Proof: Assume $(X, Y)M$ and let f be a continuous form on $X \times Y$. Define $f_1 : X \mapsto K$, $f_2 : Y \mapsto K$ by $f_1(x) = f(x, 0)$ and $f_2(x) = f(0, x)$. The forms f_1 and f_2 are continuous and can be extended continuously to E (K othe, 1969, Section 10, p. 86). We have $f(x, y) = f_1(x) + f_2(y)$ for every $(x, y) \in X \times Y$ and, by lemma 2, there exists $g \in F$ such that $g = f_1$ on X and $g = f_2$ on Y. For $(x, y) \in X \times Y$

$$g \circ T(x, y) = g(x + y) = g(x) + g(y) = f_1(x) + f_2(y) = f(x, y)$$

and therefore $f = g \circ T$.

Conversely, assume that for every continuous form f defined on $X \times Y$ there exists $g \in F$ satisfying $f = g \circ T$. Let $(f_1, f_2) \in E^2$. By definition of the product topology, $(x, y) \in X \times Y \mapsto f_1(x) \in K$ and $(x, y) \in X \times Y \mapsto f_2(y) \in K$ are continuous forms and $f = f_1 + f_2$ is continuous. There exists $g \in F$ such that $f = g \circ T$ and, for every $x \in X$,

$$f_1(x) = f(x, 0) = g \circ T(x, 0) = g(x).$$

Similarly, for $y \in Y$, $f_2(y) = g(y)$ and Lemma 2 completes the proof. □

Proposition 2. *Let (E, F) be a pair of dual spaces. A linear projection p on E is $\sigma(E, F)$-continuous if and only if* (Im p, Ker p) *is a modular pair in the lattice of all closed subspaces of E.*

proof: Assume that (Im p, Ker p)M in the lattice of all closed subspaces of E.

Let $U = \cap \mathrm{Ker}\, f_i \times \cap \mathrm{Ker}\, g_k$, $1 \le i \le m$, $1 \le k \le n$, be a neighborhoods of 0 in Im $p \times$ Ker p. Define $m + n$ forms on Im $p \times$ Ker p by $\varphi_i(x, y) = f_i(x)$ and $\gamma_k(x, y) = (x, y) = g_k(y)$. By definition of the product topology, these forms are continuous and, by using Lemma 3, there exist $\Phi_i \in F$, $1 \le i \le m$, and $\Gamma_k \in F$, $1 \le k \le n$, such that $\varphi_i = \Phi_i \circ T$, $\gamma_k = \Gamma_k \circ T$ with $T : \mathrm{Im}\, p \times \mathrm{Ker}\, p \mapsto E$ defined by $T(x, y) = x + y$.

Let $V = \cap \mathrm{Ker}\, \Phi_i \cap \bigcap \mathrm{Ker}\, \Gamma_k$. This set is a neighborhoods of 0 in E and $T(U) = V$. Therefore, T is open and p is continuous.

Conversely, let p be a continuous linear projection for the $\sigma(E, F)$ topology. Consider $f \in F$. Since $E = \mathrm{Im}\, p \oplus \mathrm{Ker}\, p$, we have $f = f \circ p + f \circ (1 - p)$ with $f \circ p \in (\mathrm{Ker}\, p)^\perp$ and $f \circ (1 - p) \in (\mathrm{Im}\, p)^\perp$. Therefore, $F =$

$(\mathrm{Im}\, p)^{\perp} + (\mathrm{Ker}\, p)^{\perp}$ and, by using Lemma 0, the pair $(\mathrm{Im}\, p, \mathrm{Ker}\, p)$ is modular.

Theorem 1. *Let L be a complete irreducible DAC-lattice. If L is representable as the lattice $\mathcal{L}$ of all closed subspaces of a pair of dual spaces (E, F) then the projection orthoposets $P(L)$ and $P(\mathcal{L})$ are isomorphic and the correspondence $p \mapsto (\mathrm{Im}\, p, \mathrm{Ker}\, p)$ is an isomorphism between the orthomodular poset of $\sigma(E, F)$-continuous linear projections defined on E onto $P(\mathcal{L})$.*

proof: It is clear that the isomorphism between the lattices L and $\mathcal{L}$ extends to an isomorphism of their projection orthoposets.

In $\mathcal{L}$, $(F, G)M^*$ is equivalent to $F + G = F \vee G$ (Mackey, 1945, Theorem III-6 or Maeda and Maeda, 1970, Lemma 31.1.5) and therefore, by Proposition 2, the correspondence $\theta : p \mapsto (\mathrm{Im}\, p, \mathrm{Ker}\, p)$ is a bijection from the set of all $\sigma(E, F)$ continuous linear projections on E onto the projection lattice $P(\mathcal{L})$ of $\mathcal{L}$. If p is continuous then so is the projection $p^{\perp} = 1 - p$ and $\mathrm{Ker}(1 - p) = \mathrm{Im}\, p$, $\mathrm{Im}(1 - p) = \mathrm{Ker}\, p$. Thus $\theta(p^{\perp}) = \theta(p)^{\perp}$ holds. For two linear projections p and q, $p \leq q$ is defined by $p = pq = qp$ and those relations hold if an only if $\mathrm{Im}\, p \subset \mathrm{Im}\, q$ and $\mathrm{Ker}\, q \subset \mathrm{Ker}\, p$. Therefore, $p \leq q \Leftrightarrow \theta(p) \leq \theta(q)$ and θ is an isomorphism of orthoposets.

Remark Let T be the lattice of all subspaces of E. The orthomodular poset $P(\mathcal{L})$ is a suborthomodular poset of $P(T)$ since, for two orthogonal elements (A, B) and (C, D) in $P(\mathcal{L})$, we have $(A, B) \vee (C, D) = (A + C, B \cap D)$.

3.1.1. The case of a locally convex space

Let E be a locally convex space and E' its topological dual. The weak topology $\sigma_s(E, E')$ is the coarsest topology on E for which all the elements of E' are continuous. Therefore this topology is coarser then the given topology on E. The weak topology $\sigma_s(E', E)$ is defined on E' in a similar way and these two weak topologies are locally convex. The topological dual of E, equipped with the $\sigma_s(E, E')$ topology, is E' and the topological dual of E', equipped with the $\sigma_s(E', E)$ topology, is E. The pair (E, E') is a dual pair and, for a subspace F of E, the following statements are equivalent (K othe, 1969, Section 20, Maeda and Maeda, 1970, Chapter VII).

- F is closed for the given locally convex topology on E;
- F is E'-closed;
- F is closed for the weak linear topology $\sigma(E, E')$.
- F is closed for the weak topology $\sigma_s(E, E')$.

If E is vector space then to every linear mapping $f : E \mapsto E$ there corresponds the adjoint mapping $f^* : E^* \mapsto E^*$, where E^* is the algebraic dual of E. If moreover E is a locally convex space then the following statements are equivalent (K othe, 1969, Section 20, 4)

- f is continuous for the weak topology;
- f is continuous for the weak linear topology;
- $f^*(E') \subset E'$.

and the following locally convex version of Theorem 1 holds.

Theorem 2. *Let E be a locally convex space and L its lattice of all closed subspaces. The projection orthoposet $P(L)$) is isomorphic to the poset of weakly continuous linear projections defined on E.*

If E is metrizable (to be a Mackey space suffices) then a linear mapping f is continuous if an only if f is weakly continuous (Schaefer, 1966, chap. IV, 7.4) and so, the orthomodular poset of projections of the lattice of all closed subspaces of a metrizable locally convex space E is isomorphic to the orthomodular poset of continuous linear projections defined on E. Normed spaces or inner product spaces are examples of metrizable locally convex spaces.

A complete metrizable locally convex space is called a Fréchet space and, in this case, all finite direct sums of closed subspaces are topological. Therefore, if L is the lattice of closed subspaces of a Fréchet space E then

$$
\begin{aligned}
P(L) &= \{(A, B) \in L^2 \mid A \cap B = \{0\},\ A \vee B = E,\ (A, B)M^*\} \\
&= \{(A, B) \in L^2 \mid A \oplus B = E\}.
\end{aligned}
$$

In this case, every linear projection with a closed kernel and a closed range is continuous.

Examples of Fréchet spaces are Banach spaces and Hilbert spaces.

Example. Let H be an inner product space, L_1 the lattice of all closed subspaces of H and L_2 the lattice of all $\perp$-closed subspaces. In general, we have only $L_2 \subset L_1$. The projection poset $P(L_1)$ is isomorphic to the orthomodular poset of continuous projections on H and $P(L_2)$ is isomorphic to the orthomodular poset of $\sigma(H, H)$ continuous projections. We recall that H is a Hilbert space if and only if $L_1 = L_2$.

In the next section, we will generalize to DAC-lattices the main idea of Ovchinnikov (1993): to determine the automorphism group of an orthoposet of projections $P(L)$ by means of automorphisms and antiautomorphisms of the lattice L.

4. AUTOMORPHISMS OF THE POSET OF PROJECTIONS OF A DAC-LATTICE

We assume that all the DAC-lattices of this section have a height ≥ 4.

In a DAC-lattice L, an element is called a finite element when it is zero or the join of a finite number of atoms. A dual finite element is a finite element of L^*.

The set of all elements which are either finite or dual finite is a complemented modular DAC-lattice denoted by $\mathcal{F}(L)$. For a finite element a of $\mathcal{F}(L)$, let us denote by $h(a)$ its height; and if a is cofinite, $h^*(a)$ is its height in L^*. If $h(a) = n$ then a has a complement b such that $h^*(b) = n$ (Maeda and Maeda, 1970, Theorem 27.10) and, for every complement c of a, $h^*(c) = n$.

We recall that for $a \in \mathcal{F}(L)$ and $b \in L$ $(a, b)M$, $(b, a)M$, $(a, b)M^*$ and $(b, a)M^*$ hold (Maeda and Maeda, 1970).

If L is a DAC-lattice and $n \in \mathbb{N}$, let us denote by $P_n(L)$ the set of all projections (a, b) of L such that a is finite and $h(a) = n$. The elements of $P_1(L)$ are the atoms of $P(L)$ and elements of $P_2(L)$ cover atoms.

For two elements (a, b) and (c, d) of $P_1(L)$ we define, as in Ovchinnikov (1993), a binary relation $\sim$ by

$$(a, b) \sim (c, d) \Leftrightarrow a = c \text{ or } b = d$$

The following lemma is the lattice version of Ovchinnikov's Lemma 3.2 in Ovchinnikov (1993). In the proof, $a \lessdot b$ means $a \leq b$ and if $a \leq c \leq b$ then $a = c$ or $c = b$.

Lemma 4. *Let L be an irreducible complete DAC-lattice. If (a, b) and (c, d) are two elements of $P_1(L)$ then $(a, b) \sim (c, d)$ if and only if $\{(a, b), (c, d)\}$ has more than one upper bound in $P_2(L)$.*

If $a \neq c$ and $b \neq d$ let $(e, f) \in P(L)$ an upper bound of $\{(a, b), (c, d)\}$. We have $e \geq a \vee c$ and $f \leq b \wedge d$. Therefore, if $h(e) = 2$ then $e = a \vee c$ and $f = b \wedge d$. The pair $\{(a, b), (c, d)\}$ has at most one upper bound in $P_2(L)$.

Conversely, assume that $(a, b) \sim (c, d)$. If $a = c$ and $b \neq d$ then $b \wedge d \lessdot b \lessdot 1$ and $b \wedge d \lessdot d \lessdot 1$. Let p and q be atoms such that $(b \wedge d) \vee p = b$ and $(b \wedge d) \vee q = d$. We have $(b \wedge d) \vee (a \vee p) = ((b \wedge d) \vee p) \vee a = b \vee a = 1$ and, as $p \leq b$, $(d \wedge d) \wedge (a \vee p) = d \wedge [(d \wedge (a \vee p)] = d \wedge [(d \wedge a) \vee p] = d \wedge p = 0$ since $p \nleq d$. Similarly, $(b \wedge d) \vee (a \vee q) = 1$ and $(b \wedge d) \wedge (a \vee q) = 0$ and thus $(a \vee p, b \wedge d)$ and $(a \vee q, b \wedge d)$ are two upper bounds of $\{(a, b), (c, d)\}$ in $P_2(L)$.

It remains to show that there exists q such that $a \vee p \neq a \vee q$. As L is complete and irreducible there exits more than one atom q such that $(b \wedge d) \vee q =$

d. If, for two different atoms q_1 and q_2, we have $(b \wedge d) \vee q_1 = (b \wedge d) \vee q_2 = d$ and $a \vee p = a \vee q_1 = a \vee q_2$ then $q_1 \vee q_2 = a \vee p$ and so

$$d = (b \wedge d) \vee q_1 \vee q_2 = (b \wedge d) \vee a \vee p = 1,$$

a contradiction. Therefore, there exists an atom q such that $(b \wedge d) \vee q = d$ and $a \vee p \neq a \vee q$.

The proof is similar if $a \neq c$, $b = d$ and if $(a, b) = (c, d)$.

Proposition 3. *(Ovchinnikov, 1993) Let L be an irreducible complete DAC-lattice and ϕ an automorphism of the poset $P(L)$. For (a, b) and (c, d) in $P_1(L)$, $(a, b) \sim (c, d)$ is equivalent to $\phi((a, b)) \sim \phi((c, d))$.*

Proof: If ϕ is an automorphism of $P(L)$ then $\phi(P_1(L)) = P_1(L)$ and $\phi(P_2(L)) = P_2(L)$. Hence, by using Lemma 0, the proposition is proved. □

Lemma 5. *Let L be an irreducible complete DAC-lattice, (a, b) and (c, d) two elements of $P_1(L)$. The following statements are equivalent.*

1. $(a, b) \not\perp (c, d)$.
2. $(a, d) \in P_1(L)$ *or* $(c, b) \in P_1(L)$.
3. *There exists* $(e, f) \in P_1(L)$ *such that* $(a, b) \sim (e, f)$ *and* $(e, f) \sim (c, d)$.

Proof: (1) ⇒ (2). If $(a, b) \not\perp (c, d)$ then $(a, b) \not\leq (d, c)$ and thus $a \not\leq d$ or $b \not\geq c$. In the first case, $(a, d) \in P_1(L)$ and in the second case $(c, d) \in P_1(L)$.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). If $a = c$ or $b = d$ then $(a, b) \not\perp (c, d)$. Otherwise, if $e = a$ then $f = d$ and if $e = b$ then $f = c$. In the first case, $a \not\leq d$ and in the second case, $b \not\geq c$ and $(a, b) \not\perp (c, d)$ holds.

By using the previous characterization of the relation ⊥ we obtain the following proposition which is the first step of the proof that every poset automorphism of $P(L)$ is actually an orthoposet automorphism. □

Proposition 4. *Let L be an irreducible complete DAC-lattice and ϕ an automorphism of the poset $P(L)$. If (a, b) and (c, d) are two elements of $P_1(L)$ then $(a, b) \perp (c, d)$ if and only if $\phi((a, b)) \perp \phi((c, d))$.*

A poset is called atomistic if every nonzero element is a join of atoms.

Lemma 6. *If L is a DAC-lattice then $P(L)$ is an atomistic poset.*

Let (a, b) be a nonzero element of $P(L)$. For every atom $p \leq a$ let q_p be a coatom such that $b \leq q_p$ and $p \wedge q_p = 0$ (Such a coatom exists since $p \not\leq b$). Similarly, for every coatom $q \geq b$ let p_q be an atom such that $p_q \leq a$ and $p_q \wedge q = 0$. Consider $X = \{(p, q_p) \mid p \text{ atom and } p \leq a\} \cup \{(p_q, q) \mid q \text{ coatom and } b \leq q\}$. We have $X \subset P_1(L)$ and (a, b) is an upper bound of X. Let (x, y) be another upper bound of X, $p \leq a$ an atom and $q \geq b$ a coatom. We have $(p, q_p) \leq (x, y)$ and $(p_q, q) \leq (x, y)$. Therefore, $p \leq x$ and $q \geq y$. Therefore, $(a, b) \leq (x, y)$ and $\bigvee X = (a, b)$.

Proposition 5. *Let P be an atomistic orthoposet and ϕ an automorphism of the poset P. If for two atoms p and q, $p \perp q$ is equivalent to $\phi(p) \perp \phi(q)$ then ϕ is an automorphism of the orthoposet P.*

Let $x \in P$ and p, q two atoms such that $p \leq \phi(x^\perp)$ and $q \leq x$. We have $\phi^{-1}(p) \leq x^\perp \leq q^\perp$ and therefore $\phi^{-1}(p) \perp q$. Hence, $p \perp \phi(q)$ and

$$\bigvee\{p \mid p \text{ atom, } p \leq \phi(x^\perp)\} \perp \bigvee\{\phi(q) \mid q \text{ atom, } q \leq x\}.$$

We have $\phi(x^\perp) \perp \phi(x)$ that is $\phi(x^\perp) \leq \phi(x)^\perp$. But $x^\perp = \phi^{-1}\phi(x^\perp) \leq \phi^{-1}(\phi(x)^\perp) \leq (\phi^{-1}\phi(x))^\perp = x^\perp$ and therefore $x^\perp = (\phi^{-1}\phi(x)^\perp)$ and $\phi(x^\perp) = \phi(x)^\perp$.

Proposition 6. *Let L be an irreducible complete DAC-lattice. Every automorphism of the poset $P(L)$ is an automorphism of the orthomodular poset $P(L)$.*

Proof: Use Lemma 6 and Propositions 4 and 5.

We denote by $At(L)$ the set of all atoms of the poset L and by $At^*(L)$ the set of all coatoms. For every $p \in At(L)$, let $p^\circ = \{(p, q) \in P_1(L)\}$ and for every $q \in At^*(L)$, ${}^\circ q = \{(p, q) \in P_1(L)\}$.

The next proposition is Theorem 2.1 of Ovchinnikov (1993) in the setting of DAC-lattices. □

Proposition 7. *Let L be an irreducible complete DAC-lattice and ϕ be an automorphism of $P(L)$. There exists a bijection $f : At(L) \mapsto At(L)$ such that, for every atom p, $\phi(p^\circ) = f(p)^\circ$ or there exists a bijection $g : At(L) \mapsto At^*(L)$ such that, for every atom p, $\phi(p^\circ) = {}^\circ g(p)$.*

Abbreviated proof. Let $p \in At(L)$ and (p, q_1), (p, q_2) be two elements of p° with $q_1 \neq q_2$. By Proposition 3, we have

$$\phi(p, q_1) = (e, f_1) \text{ and } \phi(p, q_2) = (e, f_2) \;\; \mathbf{(1)} \quad \text{or} \quad \phi(p, q_1) = (e_1, f) \text{ and } \phi(p, q_2) = (e_2, f) \;\; \mathbf{(2)}$$

When the case **(k)**, $k = 1$ or 2, is satisfied for (p, q_1) and (p, q_2) then this case is also satisfied by any similar pair of projections and so, for a given automorphism ϕ, there is only one case which occurs.

If for an automorphism ϕ the case **(1)** is satisfied then the correspondence f, defined by $f(p) = e$, is a bijection of $At(L)$ onto $At(L)$ and if it is the second case which occurs then the correspondence g such that $g(p) = f$ is a bijection of $At(L)$ onto $At^*(L)$.

Following the terminology of Ovchinnikov (1993), we say that the automorphism of $P(L)$ is even if there exists a bijection f of the set atoms of L such that, for every $(p, q) \in P_1(L)$, $\phi(p, q) = (f(p), q')$. In the other case, ϕ is called odd.

Proposition 8. *Let L be an irreducible complete DAC-lattice and ϕ an automorphism of $P(L)$.*

If ϕ is even then there exist two bijections $f_1 : At(L) \mapsto At(L)$ and $f_2 : At^*(L) \mapsto At^*(L)$ such that, for every $(p, q) \in P_1(L)$, $\phi((p, q)) = (f_1(p), f_2(q))$.

If ϕ is odd then there exist two bijections $f_1 : At(L) \mapsto At^*(L)$ and $f_2 : At^*(L) \mapsto At(L)$ such that, for every $(p, q) \in P_1(L)$, $\phi((p, q)) = (f_2(q), f_1(p))$.

Proof: Assume that ϕ is even and let $(p, q) \in P_1(L)$. By Proposition 7, there exists a bijection $f_1 : At(L) \mapsto At(L)$ such that $\phi((p, q)) = (f_1(p), q')$. As ϕ is also an even automorphism of $P(L^*)$, there exists a bijection $f_2 : At(L^*) \mapsto At(L^*)$ satisfying $\phi((q, p)) = (f_2(q), p')$ since $(q, p) \in P_1(L^*)$. But, $\phi((q, p)) = \phi((p, q))^{\perp} = (q', f_1(p))$ and therefore $\phi((p, q)) = (f_1(p), f_2(q))$. Since $At(L^*) = At^*(L)$, the proof is complete.

If ϕ is odd the proof is similar. □

Proposition 9. *Let L be an atomistic poset and f a bijection of $At(L)$. The following statements are equivalent.*

1. *There exists an automorphism $\overline{f}$ of L extending f.*
2. *For every nonzero element $x \in L$ there exists $y \in L$ such that*

$$f(\{p \in At(L) \mid p \leq x\}) = \{p \in At(L) \mid p \leq y\}$$

Proof: (1) $\Rightarrow$ (2) : clear.
(2) $\Rightarrow$ (1) Let $x \neq 0$ an element of L. The element y is unique and so we can extend f to L by means of the mapping $\overline{f}$ defined by $\overline{f}(0) = 0$ and $\overline{f}(x) = y$. Obviously, $0 \leq x$ is equivalent to $\overline{f}(0) \leq \overline{f}(x)$ and, for two non zero elements x and x', we have

$$x \leq x' \Leftrightarrow \{p \in At(L) \mid p \leq x\} \subset \{p \in At(L) \mid p \leq x'\}$$

$$\Leftrightarrow f(\{p \in At(L) \mid p \leq x\}) \subset f(\{p \in At(L) \mid p \leq x'\})$$
$$\Leftrightarrow \{p \in At(L) \mid p \leq \overline{f}(x)\} \subset \{p \in At(L) \mid p \leq \overline{f}(x')\}$$
$$\Leftrightarrow \overline{f}(x) \leq \overline{f}(x').$$

Remark that this proposition has a dual version related to bijections of the set of coatoms □

Theorem 3. *Let L be an irreducible complete DAC-lattice satisfying:*

for every $a \in L$, there exists $b \in L$ such that $(a, b) \in P(L)$. (**C**)

For every automorphism ϕ of the poset $P(L)$ there exits

1. *an automorphism f of L such that $\phi((a, b)) = (f(a), f(b))$, $(a, b) \in P(L)$, if ϕ is even,*
2. *an antiautomorphism g of L such that $\phi((a, b)) = (g(b), g(a))$, $(a, b) \in P(L)$, if ϕ is odd.*

Conversely, if f is an automorphism of L then $\phi : P(L) \mapsto L \times L$ defined by $\phi((a, b)) = (f(a), f(b))$ is an even automorphism of $P(L)$ and if g is an antiautomorphism of L then $\psi : P(L) \mapsto L \times L$ defined by $\psi((a, b)) = (g(b), g(a))$ is an odd automorphism of $P(L)$.

Sketch of the proof. Assume that ϕ is even. By Proposition 8 there exist two bijections $f_1 : At(L) \mapsto At(L)$ and $f_2 : At^*(L) \mapsto At^*(L)$ such that, for every $(p, q) \in P_1(L)$, $\phi((p, q)) = (f_1(p), f_2(q))$. By using condition (C) and Proposition 9 f_1 can be extended to an automorphism $\overline{f_1}$ of the poset L. Similarly, f_2 has an extension $\overline{f_2}$.

The correspondence $(a, b) \in P(L) \mapsto (\overline{f_1}(a), \overline{f_2}(b))$ is an automorphism of $P(L)$ which agrees with ϕ on $P_1(L)$. Therefore, for every $(a, b) \in P(L)$, $\phi((a, b)) = (\overline{f_1}(a), \overline{f_2}(b))$. This equality is also true for $(a, b) = (0, 1)$.

As ϕ is also an automorphism of $P(L)$, we have $\phi((a, b)^{\perp}) = (\phi((a, b)))^{\perp}$ that is $(\overline{f_1}(b), \overline{f_2}(a)) = (\overline{f_2}(b), \overline{f_1}(a))$ and so $\overline{f_1} = \overline{f_2}$.

The proof is similar if ϕ is odd.

For the converse, Proposition 5 shows that it suffices to prove that $(a, b) \in P(L)$ implies $(f(a), f(b)) \in P(L)$ for every automorphism f and $(g(b), g(a)) \in P(L)$ for for every antiautomorphism g. But, these implications are an easy consequences of the equivalence:

$$(a, b)M \Leftrightarrow \forall x \in L,\ ((x \wedge b) \vee a) \wedge b = (x \wedge b) \vee (a \wedge b)$$

and

$$(a, b)M^* \Leftrightarrow \forall x \in L,\ ((x \vee b) \wedge a) \vee b = (x \vee b) \wedge (a \vee b).$$

Remarque. The condition (**C**) of the previous theorem is satisfied if L is the lattice of all subspaces of a vector space or if L is an orthomodular lattice. This last case contains the case of the lattice of all closed subspaces of a Hilbert space H and in this case Theorem 3 is the main result of Ovchinnikov (1993). To replace H by a Banach space does not add new examples as a Banach space with the property that every closed subspace is complemented is isomorphic to a Hilbert space (Lindenstrauss and Tzafriri, 1971; see Plighko and Yost, 2000 for an interesting discussion about the problem of the existence of complemented closed subspaces in Banach spaces). On the other hand there exist Banach spaces with the property that the only complemented closed subspaces are the finite-dimensional or cofinite-dimensional subspaces (see Plighko and Yost, 2000 for references).

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